

THE K -THEORY LOCALIZATION OF AN UNSTABLE SPHERE

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§1. INTRODUCTION

GIVEN a space or a spectrum X , in [3, 4] Bousfield constructs a localization of X with respect to a generalized homology theory E_* (). An elegant motivation for this construction is presented in [2]. Denoting this localization by X_E , the homotopy groups of X_E should be the target of a generalized Adams spectral sequence based on E_* (). Indeed, this is shown to be true stably, in certain favorable cases, in [4].

If X is a simply connected space, or a connective spectrum (i.e., $\pi_n X = 0$ for $n < 0$), and if E_* is a connective homology theory, then X_E is just the usual arithmetic localization or completion with respect to some set of primes. If E_* () fails to be connective, then X_E is more mysterious. The case where E is periodic (real or complex) K -theory and X is a spectrum, has been studied extensively. The K -theory localization of the sphere spectrum was constructed and its homotopy groups were computed in [4]. See also [18]. In [4] the mod p homotopy groups of X_K for any spectrum X , are shown to be essentially the “mod p v_1 -periodic homotopy groups of X ”. Here v_1 refers to the Adams self-map of a Moore spectrum constructed in [1]. See [6], [10] for additional reading on stable K -localization. If X is a space, very little is known in general about the unstable K -theory localization X_K . In [17], Mislin determines the K -theory localization of Eilenberg–MacLane spaces, and proves some general arithmetic results about localization of spaces. In [5], Bousfield determines the localization of an infinite loop space with respect to K -theory. However, if X is not an infinite loop space X_K is not well understood.

In this paper we determine S_K^{2n+1} , the K -localization of an odd dimensional sphere, $n \geq 1$, that is, construct a space, show that it is the stated localization, and compute its mod p homotopy groups. By analogy with the stable result of [4], we show that the mod p homotopy groups of S_K^{2n+1} are essentially the mod p v_1 -periodic homotopy groups of S^{2n+1} . These mod p v_1 -periodic homotopy groups were defined and computed in [13], [19], [9]. It is known that for an arbitrary space X , $\pi_*(X_K; \mathbb{Z}/p)$ is not necessarily the mod p v_1 -periodic homotopy groups of X , even if X is highly connected. See [14] for a counterexample.

Let E be a spectrum representing a generalized homology theory E_* ().

Definition. A space X is called E -local if whenever a map $f: Y \rightarrow Z$ induces an isomorphism in E_* (), we have that $f^*: [Z, X] \rightarrow [Y, X]$ is a bijection. Given X , the E -localization of X is an E -local space X_E , together with a map $X \rightarrow X_E$ inducing an isomorphism in E_* ().

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It follows from the definition that X_E is unique up to homotopy. In [3] it is proved that given X and E , X_E exists. Furthermore, this construction is functorial. That is, given $f: X \rightarrow Y$, there is a map $f_E: X_E \rightarrow Y_E$ and a commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X_E & \xrightarrow{f_E} & Y_E. \end{array}$$

We now specialize to the case where $E = K$, complex, periodic K -theory. Note that by [18], localization with respect to complex and real K -theory yield the same thing. For each prime p there is a Hopf–James–Snaith map

$$s_p: QS^{2n+1} \rightarrow QD_p(S^{2n+1})$$

where $QX = \Omega^\infty \Sigma^\infty X$ and $D_p(X)$ denotes the p -adic construction $E\Sigma_p^+ \wedge_{\Sigma_p} X^{[p]}$. Each $D_p(S^{2n+1})$, localized at p , is homotopy equivalent to $\Sigma^{2n+1} B_{(n+1)q-1}$, where $B_{(n+1)q-1}$ denotes a stunted $B\Sigma_p$, localized at p , with the bottom cell in dimension $(n+1)q-1$, where $q = 2(p-1)$.

Consider the K -theory localization of the product of these maps:

$$\left(\prod_p s_p \right)_K : (QS^{2n+1})_K \rightarrow \left(\prod_p (QD_p(S^{2n+1}))_{(p)} \right)_K.$$

Let G be the homotopy fiber. Let \tilde{G} denote the simply connected cover of G . Then the composite

$$S^{2n+1} \rightarrow QS^{2n+1} \rightarrow (QS^{2n+1})_K,$$

where the first map is inclusion of the bottom cell, has a lift $S^{2n+1} \rightarrow \tilde{G}$.

THEOREM 1.1. \tilde{G} is K -local and the map $S^{2n+1} \rightarrow \tilde{G}$ induces an isomorphism in $K_*(\cdot)$. Hence $\tilde{G} = S_K^{2n+1}$.

In [13], [19] the mod p v_1 -periodic homotopy groups of a space X are defined. This definition is generalized to mod p^r v_1 -periodic homotopy groups in [9]. These periodic groups, denoted $v_1^{-1}\pi_*(X; \mathbb{Z}/p^r)$ are obtained by inverting the action of the Adams self-map of a Moore space on the mod p^r homotopy groups of X . Since the non-nilpotence of the Adams map is detected by K -theory, it is not surprising that K -theory and v_1 -periodic homotopy groups are related:

THEOREM 1.2. The localization map $S^{2n+1} \rightarrow S_K^{2n+1}$ induces an isomorphism in $v_1^{-1}\pi_k(\cdot; \mathbb{Z}/p^r)$.

By definition, if X is K -local, $v_1^{-1}\pi_k(X; \mathbb{Z}/p^r) = \pi_k(X; \mathbb{Z}/p^r)$ for k sufficiently large, hence the mod p^r homotopy groups of S_K^{2n+1} can be read off from the above theorem, Theorem 1.3 of [13], 1.2 of [19] and Theorem 1.8 of [9].

We now give the integral homotopy groups. Let B_K^{qn} denote the stable K -localization of the spectrum B^{qn} . The homotopy groups of the spectrum B_K^{qn} are computed in [8] and [7].

THEOREM 1.3.

(a) Suppose $k > 2n+1$. Then $\pi_k(S_K^{2n+1}) = \pi_{k-(2n+1)}(B_K^{qn})$

$$(b) \text{ Let } p = 2. \pi_k(S_{K(p)}^{2n+1}) = \begin{cases} 0 & k = 2n, n \equiv 1, 2 \pmod{4} \\ \mathbb{Z}/2 & k = 2n, n \equiv 0, 3 \pmod{4} \\ (\mathbb{Z}/2)^3 \oplus \mathbb{Z}_{(2)} & k = 2n+1, n \equiv 0, 3 \pmod{4} \\ \mathbb{Z}/2 \oplus \mathbb{Z}_{(2)} & k = 2n+1, n \equiv 1, 2 \pmod{4} \end{cases}$$

$$(c) \text{ Let } p > 2. \pi_k(S_{K(p)}^{2n+1}) = \begin{cases} 0 & k = 2n \\ \mathbb{Z}_{(p)} & k = 2n + 1. \end{cases}$$

§2. PROOF OF THEOREM 1.1

Let \tilde{G}_p denote the simply connected cover for the fiber of the map

$$(QS^{2n+1})_{K(p)} \rightarrow (Q\Sigma^{2n+1} B_{(n+1)q-1})_{K(p)}$$

where $K_{(p)}$ denotes p -local complex K -theory. In order to prove Theorem 1.1 it suffices to prove that $S^{2n+1} \rightarrow \tilde{G}_p$ is $K_{(p)}$ -localization for each prime p . Henceforth p will be a fixed prime.

Let F_p denote the homotopy fiber of the Snaith map

$$s: QS^{2n+1} \rightarrow Q\Sigma^{2n+1} B_{(n+1)q-1}.$$

We must compute the mod p complex K -theory of F_p . This is carried out by using J. McClure's computation of the mod p K -theory of an infinite loop space QX [15], and an Eilenberg–Moore spectral sequence argument. We recall these notions.

Historically, $K_*(QS^n; \mathbb{Z}/p)$ was first computed for $p = 2$ in [16], building on results of Hodgkin. In [15], McClure obtains a complete description of $K_*(QX; \mathbb{Z}/p)$, X a space, p a prime. For our purposes, we need:

PROPOSITION 2.1 (Miller–Snaith, McClure). *$K_*(QS^{2n+1}; \mathbb{Z}/p)$ is an exterior algebra over $K_*(pt; \mathbb{Z}/p)$ on a sequence of generators (x_1, x_2, \dots) , denoted $E(x_1, x_2, \dots)$. $K_*(Q\Sigma^{2n+1} B_{(n+1)q-1}; \mathbb{Z}/p)$ is an exterior algebra $E(y_1, y_2, \dots)$. $K_*(QS^{2n}; \mathbb{Z}/p)$ is a polynomial algebra $P(u_1, u_2, \dots)$ and $K_*(Q\Sigma^{2n} B_{(n+1)q-1}; \mathbb{Z}/p)$ is a polynomial algebra $P(v_1, v_2, \dots)$.*

These computations are made by constructing operations in the mod p K -homology of an infinite loop space analogous to the homology operations of Araki–Kudo, Browder, Dyer–Lashof and F. Cohen. More will be said about this in Section 4.

We need the Eilenberg–Moore spectral sequence in mod p K -theory. Care must be taken as to which Eilenberg–Moore spectral sequence to use, as certain ones do *not* converge. Let $F \rightarrow E \rightarrow B$ be a principal fibration. By [5], [20], there is a spectral sequence with E_2 -term given by

$$E_2^s = \text{Tor}_s^{K_*(F; \mathbb{Z}/p)}(K_*(E; \mathbb{Z}/p), \mathbb{Z}/p)$$

and converging to $K_*(B; \mathbb{Z}/p)$. We apply this to the fibration

$$Q(S^{2n}) \rightarrow Q(\Sigma^{2n} B_{(n+1)q-1}) \rightarrow F_p.$$

We need 2.1 and the following lemma whose proof is given in Section 4.

LEMMA 2.2. *The map*

$$\Omega_S: QS^{2n} \rightarrow Q\Sigma^{2n} B_{(n+1)q-1}$$

induces an algebra homomorphism in $K_(; \mathbb{Z}/p)$*

$$P(u_1, u_2, \dots) \xrightarrow{(\Omega_S)_*} P(v_1, v_2, \dots)$$

satisfying:

- (1) $(\Omega s)_*(u_1) = 0$.
- (2) $P(u_2, u_3, \dots)$ is mapped isomorphically onto $P(v_1, v_2, \dots)$.

This yields

COROLLARY 2.3. *The inclusion $S^{2n+1} \rightarrow QS^{2n+1}$ lifts to a map $S^{2n+1} \rightarrow F_p$ which induces an isomorphism $K_*(S^{2n+1}; \mathbb{Z}_{(p)}) \xrightarrow{\cong} K_*(F_p; \mathbb{Z}_{(p)})$.*

Proof. By 2.1 the E_2 -term of the spectral sequence is given by

$$\mathrm{Tor}_s^{P(u_1, u_2, \dots)}(P(v_1, v_2, \dots); \mathbb{Z}/p)$$

where the action of $P(u_1, \dots)$ on $P(v_1, v_2, \dots)$ is given by 2.2. It follows easily that this Tor group is isomorphic to an exterior algebra on one generator. This leaves no room for differentials and we conclude that the map induces an isomorphism in mod p K -theory. Since $D_j(S^{2n+1})$ is torsion for $j > 1$ the map is also an isomorphism in rational homology. The conclusion follows. ■

The next step in the proof of Theorem 1 is to determine the $K_{(p)}$ -localization of F_p (which is the same as that of S^{2n+1} by 2.3). This can be done since F_p is in a fibration involving infinite loop spaces, and in [5], Bousfield determines the $K_{(p)}$ -localization of an infinite loop space. Roughly speaking, let X be a connective spectrum and let $\Omega^\infty X$ denote the 0th space of an associated Ω -spectrum. Then $(\Omega^\infty X)_{K_{(p)}}$ is "almost" $\Omega^\infty(X_{K_{(p)}})$. By "almost" we mean the two spaces have the same homotopy groups except in the bottom few dimensions. More specifically,

THEOREM 2.4 (Theorem 3.1 of [5]). *For a connective, p local spectrum X there are natural isomorphisms*

$$\begin{aligned} \pi_i((\Omega^\infty X)_{K_{(p)}}) &\cong \pi_i X_{K_{(p)}}, & i > 2 \\ \pi_i((\Omega^\infty X)_{K_{(p)}}) &\cong \pi_i X, & i < 2 \end{aligned}$$

and a natural short exact sequence

$$0 \rightarrow \mathrm{tors} \pi_2(X_{K_{(p)}}) \rightarrow \pi_2((\Omega^\infty X)_{K_{(p)}}) \rightarrow \frac{\pi_2(X)}{\mathrm{tors} \pi_2(X)} \rightarrow 0.$$

This result should be seen in the light of the fact that for a spectrum X , the stable localization $X_{K_{(p)}}$ is very well understood.

Let G_p be the homotopy fiber of the localized Snaith map

$$s_{K_{(p)}}: (QS^{2n+1})_{K_{(p)}} \rightarrow (Q\Sigma^{2n+1} B_{(n+1)q-1})_{K_{(p)}}.$$

By 12.9 of [3], the homotopy fiber of a map between $K_{(p)}$ -local spaces is $K_{(p)}$ -local. Hence G_p is $K_{(p)}$ -local. However, the calculation yielding $K_*(F_p; \mathbb{Z}/p)$ does not quite yield $K_*(G_p; \mathbb{Z}/p)$. The difficulty lies in the fact that $\Omega((QS^{2n+1})_{K_{(p)}})$ is not the same as $(QS^{2n})_{K_{(p)}}$. This difficulty is surmounted by passing to connected covers. If X is a space, let \tilde{X} denote its 1-connected cover.

LEMMA 2.5. *Let $X = \Omega((QS^{2n+1})_{K_{(p)}})$. Then \tilde{X} is equivalent to $(QS^{2n})_{K_{(p)}}$. Similarly, let $Y = \Omega((Q\Sigma^{2n+1} B_{(n+1)q-1})_{K_{(p)}})$. Then \tilde{Y} is equivalent to $(Q\Sigma^{2n} B_{(n+1)q-1})_{K_{(p)}}$.*

Proof. Since $\Omega((QS^{2n+1})_{K(p)})$ is $K(p)$ -local, loops on the localization map

$$QS^{2n} = \Omega QS^{2n+1} \rightarrow \Omega((QS^{2n+1})_{K(p)})$$

extends to a map $(QS^{2n})_{K(p)} \rightarrow \Omega((QS^{2n+1})_{K(p)})$.

Apply Theorem 2.4 to conclude that this extension induces an isomorphism in $\pi_i(\)$ for $i > 2$, noting that in the stable category, localization commutes with suspension. This extension induces an isomorphism in π_2 again by 2.4, noting that $\pi_i^s(S_{K(p)}^0)$ is torsion unless $i = 0$, in which case

$$\pi_0(S_{K(p)}^0) = \text{tors}(\pi_0 S_{K(p)}^0) \oplus \mathbb{Z}_{(p)}$$

(see page 393 of [18]). Finally, by 2.8 of [17], $(QS^{2n})_{K(p)}$ is 1-connected. The conclusion follows. A similar argument applies to $B_{(n+1)q-1}$. ■

Let $\varphi = \ker[\pi_2(QS^{2n+1})_{K(p)} \rightarrow \pi_2(Q\Sigma^{2n+1}B_{(n+1)q-1})_{K(p)}]$ and let \tilde{G}_p be the fiber of the map $\tilde{G}_p \rightarrow K(\varphi, z)$, we have a diagram of principal fibrations by 2.5:

$$\begin{array}{ccccc} (QS^{2n}) & \xrightarrow{\Omega s} & Q(\Sigma^{2n}B_{(n+1)q-1}) & \rightarrow & F_p \\ \downarrow & & \downarrow & & \downarrow \\ (QS^{2n})_{K(p)} & \longrightarrow & (Q\Sigma^{2n}B_{(n+1)q-1})_{K(p)} & \rightarrow & \tilde{G}_p. \end{array} \quad (2.7)$$

This induces a map of Eilenberg–Moore spectral sequences which is an isomorphism on E_2 terms. This implies that the right-hand vertical map in (2.7) induces an isomorphism in $K_*(\ ; \mathbb{Z}/p)$. Applying the spectral sequence to the fibration $K(\varphi, 1) \rightarrow \tilde{G}_p \rightarrow \tilde{G}_p$, one sees that the map $\tilde{G}_p \rightarrow \tilde{G}_p$ induces an isomorphism in $K_*(\ ; \mathbb{Z}/p)$. Thus $S^{2n+1} \rightarrow \tilde{G}_p$ induces an isomorphism in mod p K -theory as well. This map is clearly a rational equivalence, hence induces an isomorphism in $K_*(\ ; \mathbb{Z}_{(p)})$. Since \tilde{G}_p is the fiber of a map $G_p \rightarrow K(\pi_1 G_p, 1)$ and $K(\pi_1 G_p, 1)$ is $K(p)$ -local by 2.2 of [17], \tilde{G}_p is $K(p)$ -local. This completes the proof of Theorem 1. ■

§3. PROOF OF THEOREMS 1.2 AND 1.3

Recall that G_p is the fiber of the localized map

$$(QS^{2n+1})_{K(p)} \xrightarrow{s_{\Lambda, \infty}} (Q\Sigma^{2n+1}B_{(n+1)q-1})_{K(p)}.$$

We loop this map $2n+1$ times and apply $\pi_i(\)$. By [11], before localizing the map $\Omega^{2n+1}s$ factors as:

$$\begin{array}{ccc} QS^0 & \xrightarrow{\Omega^{2n+1}s} & QB_{(n+1)q-1} \\ & \searrow s' & \nearrow \rho \\ & QB & \end{array}$$

where s' is a Snaith map and ρ is just the pinch map. After localizing, the map $(QS^0)_{K(p)} \xrightarrow{s'_{\Lambda, \infty}} (QB)_{K(p)}$ induces an isomorphism in $\pi_i(\)$ for $i > 0$. By Theorem 2.4 the homotopy groups of the fiber of the map $(QB)_{K(p)} \xrightarrow{\rho_{\Lambda, \infty}} (QB_{(n+1)q-1})_{K(p)}$ are the homotopy groups of the spectrum $B_{K(p)}^{qn}$. This proves:

PROPOSITION 3.1. $\pi_{i+2n+1}(G_p) \cong \pi_i(B_{K(p)}^{qn})$ for $i \geq 1$.

The homotopy groups of $B_{K(p)}^{qn}$ are computed, for example, in [8] and in [7]. This is the first part of Theorem 1.3.

The homotopy groups of the spectra $B_{K(p)}$ and $(B_{(n+1)q-1})_{K(p)}$ are also given in [7], [8]. The homotopy groups of the spectrum $S_{K(p)}^{2n+1}$ are computed in [4], [18]. The remaining values of $\pi_i(S_{K(p)}^{2n+1})$ given in the theorem are thus read off from the long exact sequence of the fibration

$$G_p \rightarrow (QS^{2n+1})_{K(p)} \rightarrow (Q\Sigma^{2n+1} B_{(n+1)q-1})_{K(p)}. \quad \blacksquare$$

To prove Theorem 1.2, recall from [13], [19], [9] that there is a Snaith map

$$\Omega^{2n+1} S^{2n+1} \rightarrow QB^{qn}$$

which is shown to induce an isomorphism in $v_1^{-1}\pi_*(\ ; \mathbb{Z}/p^r)$. This map is compatible with the map $s': QS^0 \rightarrow QB$ of the preceding paragraphs. Thus the isomorphism

$$v_1^{-1}\pi_{i+2n+1}(S^{2n+1}; \mathbb{Z}/p^r) \rightarrow v_1^{-1}\pi_{i+2n+1}(S_{K(p)}^{2n+1}; \mathbb{Z}/p^r)$$

for $i \geq 1$ follows from 1.3 of [13], 1.2 of [19] and Proposition 3.1. Finally, the result for all $v_1^{-1}\pi_k(\)$ follows from periodicity.

§4. PROOF OF 2.2.

In [15], an operation Q is constructed which goes from the mod p^{r+1} K -homology of $QX = \Omega^c \Sigma^c X$ to the mod p^r K -homology of QX for a space X . For example, $K_*(QS^{2n+1}; \mathbb{Z}/p) = E(x_1, x_2, \dots)$ and the sequence $\{x_i\}$ is obtained by alternately lifting to $K_*(QS^{2n+1}; \mathbb{Z}/p^2)$ and applying the operation Q , beginning with x_1 , which is the generator of $K_*(S^{2n+1}; \mathbb{Z}/p)$. Likewise the other three sequences of generators $\{y_i\}$, $\{u_i\}$, and $\{v_i\}$ of 2.1 are obtained this way. Define a weight filtration in $K_*(QX; \mathbb{Z}/p)$ by assigning elements in $K_*(X; \mathbb{Z}/p^r)$ weight 1, $\text{weight}(Qx) = p \cdot \text{weight}(x)$, and $\text{weight}(xy) = \text{weight}(x) + \text{weight}(y)$. There is a stable splitting $\Sigma^c QX = \Sigma^c \bigvee_{j \geq 1} D_j(X)$, where $D_j(X)$ is the j -adic construction. Theorem 4.1 of [15] then says that an additive basis for $K_*(D_k(X); \mathbb{Z}/p)$ is given by the monomials in $K_*(QX; \mathbb{Z}/p)$ of weight k .

Recall from [15] the construction of the extended power spectrum $D_\pi X$, where $\pi \subset \Sigma_j$. If $\pi = \Sigma_j$ we just have $D_{\Sigma_j} X = D_j(X)$. This is functorial in π in the sense that if $\rho \subset \pi$ there is an induced map $D_\rho X \rightarrow D_\pi X$. There is also a transfer $D_\pi X \rightarrow D_\rho X$. The iterated construction $D_{\Sigma_r}(D_{\Sigma_q}(X))$ is just $D_\pi(X)$ where π is the wreath product $\Sigma_r \wr \Sigma_q$.

We use a result of N. Kuhn:

PROPOSITION 4.1 ([12]). Consider the composite map

$$\Sigma^c D_m(X) \rightarrow \Sigma^c QX \rightarrow \Sigma^c QD_\rho(X) \rightarrow \Sigma^c D_r(D_\rho(X)) \quad (4.2)$$

- (1) If $r < m/p$ then this map is nullhomotopic.
- (2) If $m = rp$ then this map is homotopic to the transfer associated to the inclusion $\Sigma_r \wr \Sigma_p \subset \Sigma_m$.

Proof of 2.2. $(\Omega s)_*(u_1) = 0$ since u_1 is in the image of

$$K_*(S^{2n}; \mathbb{Z}/p) \rightarrow K_*(QS^{2n}; \mathbb{Z}/p)$$

and $S^{2n} \rightarrow QS^{2n} \rightarrow Q\Sigma^{2n} B_{(n+1)q-1}$ is null. This proves the first part of 2.2.

For polynomials in u_i , $i \geq 2$, consider the diagram

$$\begin{array}{ccc} \Omega QS^{2n+1} & \xrightarrow{\Omega s} & \Omega QD_\rho S^{2n+1} \\ \uparrow = & & \uparrow \rho \\ QS^{2n} & \xrightarrow{s} & QD_\rho S^{2n} \end{array}$$

By 2.1 $K_*(D_p S^{2n}; \mathbb{Z}/p) = \mathbb{Z}/p \oplus \mathbb{Z}/p$ generated by u_2 and u_1^p . Hence by [15] $K_*(QD_p S^{2n}; \mathbb{Z}/p)$ is a polynomial algebra on sequences $\{v_1, v_2, \dots\}$ and $\{w_1, w_2, \dots\}$. The sequence $\{v_i\}$ is obtained by iterating the operation Q on u_2 and the sequence $\{w_i\}$ is obtained by iterating the operation Q on u_1^p . The map ρ is just Q applied to the pinch map, hence in K -theory it induces the projection onto $P(v_1, v_2, \dots)$ given by modding out by the ideal generated by $\{w_i\}$. Call this ideal W . We consider 4.2 where $X = S^{2n}$.

Let f be a polynomial in $P(u_1, \dots)$ of weight m and suppose m is divisible by p . Hence f is actually in $P(u_1^p, u_2, \dots)$. First note that by 4.1 (1) the component of $s_*(f)$ for $r < m/p$ is 0. Let g_f denote the polynomial in $P(w_1, v_1, v_2, \dots)$ obtained from f by replacing u_i with v_{i-1} for $i \geq 2$ and u_1^p by w_1 . We collect some facts about g_f . Let $D_{m/p} D_p(S^{2n}) \xrightarrow{i} D_m(S^{2n})$ be the map induced by the inclusion of the wreath product.

LEMMA 4.3.

- (1) $i_*(g_f) = f$.
- (2) The kernel of i_* is generated by w_i for $i \geq 2$.
- (3) $g_{(i_*, h)} \equiv h \pmod{W}$ for all h .

Proof of 4.3. Part (1) is clear. Part (2) follows from 3.3 (vii) of [15] (see also 3.7 (ix) of [15]). For part (3), we have $i_* g_{(i_*, h)} = i_* h$ by part (1), the result follows by (2). ■

Now consider the composite

$$D_m(S^{2n}) \xrightarrow{\tau} D_{m/p} D_p(S^{2n}) \xrightarrow{i} D_m(S^{2n}) \quad (4.4)$$

where the first map is the transfer and the second map is as above.

LEMMA 4.5. Up to a unit in \mathbb{Z}/p , the component of $s_*(f)$ for $r = m/p$ has the form $g_f + \phi_f$ where ϕ_f has the property that $i_*(\phi_f)$ has Atiyah-Hirzebruch filtration strictly less than that of f .

Proof of 4.5. By 4.1 (2) the component of $s_*(f)$ of weight $r = m/p$ is given by the transfer. Since $\Sigma_{m/p} \Sigma_p$ has index prime to p in Σ_m the composite 4.4 induces multiplication by k in mod p homology, where k is some unit in \mathbb{Z}/p . Thus the induced isomorphism of the E_∞ -terms of Atiyah-Hirzebruch spectral sequences is multiplication by k . Thus the composite 4.4 in mod p K -theory sends f to $kf + \psi_f$ where ψ_f has Atiyah-Hirzebruch filtration less than f . Let ϕ_f be $s_*(f) - kg_f$ and the lemma follows using 4.3 (1). ■

In order to consider values of r larger than m/p we make a definition. Note that if r is such that the connectivity of $D_r D_p S^{2n}$ is greater than the Atiyah-Hirzebruch filtration of f then the component of $s_*(f)$ of weight r is 0. Define the *range* of f , denoted $R(f)$, to be $R - m/p$, where R is the largest integer such that the connectivity of $D_R D_p S^{2n}$ is less than or equal to the Atiyah-Hirzebruch filtration of f .

The proof of 2.2 is based on the following:

LEMMA 4.6. If $f \in P(u_1^p, u_2, \dots)$ with weight m , then g_f is in the image of s_* modulo the ideal W .

Proof of 4.6. The proof is by double induction on the range of f and the Atiyah-Hirzebruch filtration of f . The statement is clear if $R(f) = 0$ and the filtration of f is

minimal. Suppose it is true for all f with range smaller than N , and for all f with range equal to N and filtration smaller than M . Suppose we are given an f with $R(f) = N$ and filtration M . Let h be the term of $s_*(f)$ of weight r , for some r such that $r > m/p$. Since the Atiyah–Hirzebruch filtration of $i_*(h)$ is less than or equal to that of h , which is less than or equal to that of f , and since the weight of $i_*(h)$ is greater than that of f , we have $R(i_*(h)) < R(f)$. By the induction hypothesis, $g_{i_*(h)}$ is in the image of $s_* \bmod W$. But $g_{i_*(h)} = h \bmod W$ by 4.3 (3). Since h was any term of higher weight in $s_*(f)$ we conclude that $g_f + \phi_f$ is in the image of $s_* \bmod W$.

Now $i_*(\phi_f)$ has weight equal to f and Atiyah–Hirzebruch filtration smaller. By induction, $g_{i_*(\phi_f)}$ is in the image of $s_* \bmod W$, but $g_{i_*(\phi_f)} = \phi_f \bmod W$, so ϕ_f is in the image of $s_* \bmod W$. Therefore g_f is in the image of $s_* \bmod W$. ■

Returning to the Proof of 2.2 (2), apply Lemma 4.6 to the case where $f = u_i$, $i \geq 2$ to conclude that v_i , $i \geq 1$ is in the image of s_* , modulo W , i.e. that Ωs_* is surjective.

To prove the injectivity part of 2.2 (2) we use the following lemma:

LEMMA 4.7. *Modulo decomposables, the set $\{\Omega s_*(u_i) \mid i \geq 2\}$ is a vector space basis for the vector space spanned by $\{v_i \mid i \geq 1\}$.*

Proof. There is a homology suspension

$$\sigma_*: K_i(\Omega X; \mathbb{Z}/p) \rightarrow K_{i+1}(X; \mathbb{Z}/p)$$

and McClure's operation is compatible with this suspension (3.3 [15]). In the notation of Section 2, we have

$$\sigma_* u_i = x_i, \quad i \geq 1$$

and

$$\sigma_* v_i = y_i, \quad i \geq 1.$$

We have a commutative diagram

$$\begin{array}{ccc} QS^{2n+1} & \xrightarrow{\sigma} & QD_p S^{2n+1} \\ \uparrow \sigma & & \uparrow \sigma \\ \Sigma QS^{2n} & \xrightarrow{\Sigma \Omega s} & \Sigma \Omega QD_p S^{2n+1} \end{array}$$

Since the homology suspension annihilates decomposables, $\sigma_* \Omega s_*(u_i)$ is some linear combination of the elements $\{y_j \mid j \geq i-1\}$. But this is equal to $s_* \sigma_*(u_i) = s_*(x_i)$. Applying 4.1(2) when $X = S^{2n+1}$ we conclude that the coefficient of y_{i-1} in $s_*(x_i)$ is nonzero. Using the same argument as in the Proof of 4.6 we conclude that the set $\{s_*(x_i) \mid i \geq 2\}$ is a vector space basis for the vector space spanned by $\{y_i \mid i \geq 1\}$. The lemma now follows. ■

The injectivity of the map $\Omega s_*: P(u_2, \dots) \rightarrow P(v_1, \dots)$ now follows immediately from 4.7 and this completes the Proof of 2.2. ■

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